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Attractors and Characteristic Exponents

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A natural definition of an attractor as an invariant measure is given (based on the ergodic theory of axiom A diffeomorphisms) and some results are proved which support this definition. It is also proved that if an attractor has every characteristic exponent less than zero in a set of nonzero measure, then the support set of the attractor is an asymptotic stable periodic orbit.

KEY WORDS: Attractors; ergodic theory; characteristic exponents.

1. INTRODUCTION

A physically reasonable definition of "attractor" is necessary if this definition is to be put to a practical test. Milnor[6] has an interesting discussion of some of the difficulties related to the attractor concept.

The characteristic exponents (CEs) have been extensively used, in most cases in an intuitive manner, to characterize attractors. The CE are important because they can be estimated in computer experiments and from experimental signals.[3,4,9,10] Although the CEs are defined in a set of total invariant (probability) measure, if we want to use them to characterize the attractors it is necessary to associate an invariant measure to each attractor.

In Section 2, based on results of the ergodic theory of C^2 axiom A diffeomorphisms, a natural definition is given of an attractor as an invariant measure and some results are proved which support this definition.

Section 3 studies the relations between attractors and CEs; if an attractor has all CEs less than zero in a set of nonzero measure, then the

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support set of the "attractor" is an asymptotic stable periodic orbit. The proof of this result is given in Section 4.

Section 5 gives a discussion of the case of flows.

2. ATTRACTORS

In this work X will denote a smooth compact manifold. The Riemannian structure of X induces a volume measure l on X, and this measure will be called a Lebesgue measure. It does not really matter which particular measure is used, since I only distinguish between sets of Lebesgue measure zero and sets of positive Lebesgue measure.

Let $C(X)$ denote the Banach algebra of real, continuous functions on X with the uniform norm. Recall that the set $M(X)$ of probability measures on X forms a convex, compact, metrizable subset of the weak dual of $C(X)$ (the topology of the weak dual is called the weak*-topology).

Let $\mathscr{A}: X \to X$ be a continuous map. The set $I(\mathscr{A})$ of $\mathscr{A}$-invariant probability measures is a nonempty convex and compact subset of $M(X)$. I shall use a metric d compatible with the topology of X, but the results will not depend on the special choice of d. I shall also use the induced map $\mathscr{A}_*: M(X) \to M(X)$ given by $(\mathscr{A}_* \mu)(S) = \mu(\mathscr{A}^{-1} S)$; note that $\mathscr{A}_* \delta_x = \delta_{\mathscr{A} x}$, where δ_x is the unit mass at $x \in X$.

Recall some results of the ergodic theory of C^2 axiom A diffeomorphisms.[1] Let $f: X \to X$ be a C^2 axiom A diffeomorphism and Ω a basic set that has a neighborhood U with $f(U) \subset U$; then there exist a set W and an ergodic measure σ such that $U \setminus W$ has zero Lebesgue measure, and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f^i_* \delta_x = \sigma \qquad \text{(weak*-topology)}$$

whenever $x \in W$.

The following definitions are based on the above results.

Definition. The measure $\nu \in I(\mathscr{A})$ attracts $\mu \in M(X)$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathscr{A}^i_* \mu = \nu$$

Definition. Let $\nu \in I(\mathscr{A})$. The set $G_\nu = \{x \in X: \nu \text{ attracts } \delta_x\}$ is called the attracting set of ν.

Definition. The measure $\nu \in I(\mathscr{A})$ is an attractor for $\mathscr{A}: X \to X$ if $l(G_\nu) > 0$.

Now, I prove some results that support the above definitions. The integral $\int_X f\, d\mu$ will be denoted by $\int f\, d\mu$ or $\mu(f)$, and $\sum_{i=0}^{n-1} f(\mathscr{A}^i x)$ by $(S_n f)(x)$.

Proposition 1. G_μ is a measurable set for any $\mu \in I(\mathscr{A})$.

Proof. If $f \in C(X)$, we define $\Sigma_f\colon X \to R$ and $\sigma_f\colon X \to R$ by

$$\Sigma_f(x) = \lim_{n\to\infty} \sup \frac{1}{n} (S_n f)(x)$$

$$\sigma_f(x) = \lim_{n\to\infty} \inf \frac{1}{n} (S_n f)(x)$$

Since $(S_n f)/n$ is continuous for each $n \in N$, we have that Σ_f and σ_f are measurable functions.

Let Λ_f be the subset of points x of X such that $\lim_{n\to\infty}(1/n)(S_n f)(x)$ exists and is equal to $\mu(f)$. This set consists of $\Sigma_f^{-1}(\mu(f)) \cap \sigma_f^{-1}(\mu(f))$, so Λ_f is a measurable set, since each set in this intersection is measurable.

Choose a countable dense subset $(f_k)_1^\infty$ of $C(X)$. By approximating a given $g \in C(X)$ by members of $(f_k)_1^\infty$, it follows that $G_\mu = \bigcap_{k\geqslant 1} \Lambda_{f_k}$. Since each Λ_{f_k} is measurable, we get that G_μ is measurable. ∎

Proposition 2. The number of distinct attractors for $\mathscr{A}$ is at most countably infinite.

Proof. It is sufficient to note that if the measures ν and μ are distinct, then $G_\nu \cap G_\mu = \varnothing$. ∎

Milnor[6] has proposed a different definition of an attractor based on the privileged role which the omega limit set $\omega(x)$ of a point $x \in X$ should play for the asymptotic behavior of dynamical systems. Recall that $\omega(x)$ is the collection of all accumulation points for the sequence $(x, \mathscr{A}x, \mathscr{A}^2x,\ldots)$, and that $\omega(x)$ is always closed and nonempty.

Proposition 3. If $\mu \in I(\mathscr{A})$ attracts δ_x, then $\mu(\omega(x)) = 1$.

Proof. Let $x \in G_\mu$. Note that $\omega(x) = \bigcap_{n\geqslant 0} \overline{O(\mathscr{A}^n x)}$, where $O(z)$ is the orbit of $z \in X$ defined as $O(z) = \{z, \mathscr{A}z, \mathscr{A}^2 z,\ldots\}$, and the bar represents the closure operation.

If $O(x)$ is dense in X, the same occurs with $O(\mathscr{A}^n x)$ $\forall n \in N$, and clearly $\omega(x) = X$. If $O(x)$ is not dense in X, then $X \setminus \overline{O(x)}$ is a nonempty open set.

Let $\varepsilon > 0$ be small enough for

$$K = \overline{\{y \in X\colon d(y, \overline{O(x)}) > \varepsilon\}}$$

to be nonempty. Consider the Urysohn function

$$u_\varepsilon(y) = \begin{cases} 1 & \text{if } y \in \overline{O(x)} \\ 0 & \text{if } y \in K \\ 0 < u_\varepsilon < 1 & \text{otherwise} \end{cases}$$

Since u_ε is continuous, we have

$$1 = u_\varepsilon(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} u_\varepsilon(\mathscr{A}^i x) = \int u_\varepsilon \, d\mu$$

Hence $\int u_\varepsilon \, d\mu = 1$ and then $\mu(\overline{O(x)}) = 1$. Similarly one proves that $\mu(\overline{O(\mathscr{A}^n x)}) = 1$ for any $n \in N$. The proposition follows readily from $\omega(x) = \bigcap_{n \geqslant 0} \overline{O(\mathscr{A}^n x)}$. ▮

If μ attracts δ_x, one can conclude, by Proposition 3, that μ gives a probability distribution on $\omega(x)$ and also $\mu(X \setminus \omega(x)) = 0$, so the knowledge of μ is physically more important than that of $\omega(x)$ itself.

In general it is easier to work with the Lebesgue measure than with the invariant measure.

Proposition 4. Let $\nu \in I(\mathscr{A})$ be an attractor for $\mathscr{A}$; then there are absolutely continuous measures with respect to Lebesgue measure l attracted by ν.

Proof. Let $h \in L^1(l)$ be a positive function such that $Y = \int_{G_\nu} h \, dl > 0$. Define the measure $\mu \in M(X)$ by

$$d\mu = (\chi_{G_\nu} h)/Y \, dl$$

If $f \in C(X)$, then

$$\begin{aligned} \int f \, d\left(\frac{1}{n} \sum_{i=0}^{n-1} \mathscr{A}^i_* \mu\right) &= \int \frac{(S_n f)}{n} \, d\mu \\ &= \int \frac{1}{n} \frac{(f \circ \mathscr{A}^i) \, \chi_{G_\nu}}{Y} \, dl \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \left[\int \frac{(f \circ \mathscr{A}^i) \, \chi_{G_\nu} h}{Y} \right] dl \end{aligned}$$

Since $\mu(G_\nu) = 1$, one can apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty}\int f\,d\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathscr{A}_*^i\mu\right)=\int_{G_\nu}\lim_{n\to\infty}\frac{(S_nf)}{n}\frac{h}{Y}\,dl$$

$$=\int_{G_\nu}\left(\int f\,d\nu\right)d\mu=\int f\,d\nu$$

Since $f\in C(X)$ is arbitrary, the proof is complete. ∎

This work has involved the assumption that attractors, which would be observed in practice, are associated with time averages of initial conditions in a set of nonzero Lebesgue measure. A general proof of the existence of time averages is an old problem; Birkhoff's ergodic theorem proves the existence of such averages in a set of total invariant measures, but in general, the invariant measures are singular with respect to Lebesgue measure.

3. CHARACTERISTIC EXPONENTS

The characteristic exponents (CEs) are well defined in a set of total invariant measure by a theorem of Oseledec.[4,8] Denote the CEs at a point $x\in X$ by $\lambda^1(x)\geqslant\lambda^2(x)\geqslant\lambda^3(x)\geqslant\cdots$; recall that if $\mu\in I(\mathscr{A})$ is ergodic, then the CEs are constant μ-a.e.

As already observed, it is possible to estimate the CEs from experimental signals and in computer experiments, so the use of the CEs would be an interesting way to try to characterize attractors; also of interest are the intuitive relations between positive CEs and "chaos."

Unfortunately, knowledge of the CEs (in most cases this is the information we have in an experiment) is not enough, in general, to characterize the attractors. There are examples of ergodic attractors μ such that $\lambda^1>0$ μ-a.e., some of them with nonzero metric entropy[4] and others concentrated on a periodic orbit. One example of the last case is given by the map $f\colon[-a,a]\to[-a,a]$, $a=(64/5\sqrt{5})$,

$$f(x)=\begin{cases}4x(1-x^2)^2 & \text{if}\quad x\in[-1,1]\\ 0 & \text{otherwise}\end{cases}$$

where δ_0 is the unique attractor and $\lambda^1(0)=\log 4$.

Definition. An *asymptotic stable periodic orbit* (ASPO) is a periodic orbit P that has a neighborhood U such that $d(\mathscr{A}^nx,P)\to 0$ as $n\to\infty$ for any $x\in U$.

If $\mu\in I(\mathscr{A})$ is an attractor such that $\lambda^1=0$ μ-a.e., many possibilities may occur: the support set of μ could be an ASPO, a quasiperiodic orbit on a torus, something like the "Feigenbaum attractor,"[4,6] etc.

In the case where $\lambda^1 < 0$ a.e. the following result due to Ruelle[8] holds.

Lemma 5. Let $\mathscr{A}: X \to X$ be a $C^{1+\alpha}$ map. If $\mu \in I(\mathscr{A})$ is ergodic and all CEs are less than zero μ-a.e., then the support of μ is an ASPO.

In practice, it is very hard to verify if an observed attractor is ergodic or not, so a generalization of Lemma 5 which does not suppose ergodicity would be welcome. One can easily imagine nonergodic measures such that all CEs are less than zero a.e.

Theorem 6. Let ν be an invariant measure for the $C^{1+\alpha}$ map $\mathscr{A}: X \to X$ such that the following conditions hold:

(i) $G_\nu \neq \varnothing$.

(ii) There is a measurable set $F \subset X$ of nonzero ν-measure such that all CEs are less than zero at any point of F.

Then ν is ergodic and its support is an ASPO.

Corollary. If instead of (i) in Theorem 6 it is assumed that (i′) *ν is an attractor for $\mathscr{A}$*, then the same conclusions hold.

4. PROOF OF THEOREM 6

For the proof of Theorem 6 we need the ergodic decomposition theorem. Let K be the set of points of X such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathscr{A}^i_* \delta_x$$

exists; it easily follows by the Riesz representation theorem[5] that for each $x \in K$ there is a unique $\mu_x \in I(\mathscr{A})$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathscr{A}^i_* \delta_x = \mu_x$$

Let J denote the set $\{x \in K\colon \mu_x \text{ is ergodic and } x \in \text{support } \mu_x\}$.

Lemma 7 (Ergodic decomposition). If $\mu \in I(\mathscr{A})$, then $\mu(J) = 1$, any $g \in L^1(\mu)$ is μ_x-integrable for x in a subset of J of total μ-measure, and

$$\int \left(\int g \, d\mu_x \right) d\mu(x) = \int g \, d\mu$$

The proof of Lemma 7 can be found in the book by Mañé.[5]

Proof of Theorem 6. First note that $\int_F \lambda^1 \, d\nu < 0$. By the ergodic decomposition of ν (Lemma 7), one has

$$\int_F \lambda^1 \, d\nu = \int \chi_F \lambda^1 \, d\nu = \int \left(\int_F \lambda^1 \, d\mu_y \right) d\nu(y) < 0$$

Thus, there is a measurable set T with $\nu(T) > 0$ and $\int_F \lambda^1 \, d\mu_y < 0$ whenever $y \in T$.

Lemma 8. The cardinality of T is at most countably infinite, and if $y \in T$, the support of μ_y is an ASPO.

Proof. Let $y \in T$. Since $\int_F \lambda^1 \, d\mu_y < 0$, one has $\lambda^1(z) < 0$ in a set of nonzero μ_y-measure. One has $\lambda^1 < 0$ μ_y-a.e., as it follows from the ergodicity of μ_y. By Lemma 5 the support of μ_y is an ASPO ($y \in T$).

Define $\phi: T \to I(\mathscr{A})$ by $\phi(w) = \mu_w$. Since w belongs to the support of μ_w and this support is an ASPO, it follows that the cardinality of $\phi^{-1}(\mu_w)$ is finite. Clearly, there are at most countably infinite ASPOs; therefore, the cardinality of T is also at most countably infinite. ∎

Now, we have the decomposition

$$\nu = \sum_{y \in T}{}' q_y \mu_y + q\sigma$$

in the sense that

$$\nu(f) = \sum_{y \in T}{}' q_y \mu_y(f) + q\sigma(f) \quad \forall f \in C(X)$$

where $\sum'$ denotes sum without repetition in $\mu_y (y \in T)$ and q_y, q are nonnegative real numbers such that $\sum'_{y \in T} q_y > 0$ and $q + \sum'_{y \in T} q_y = 1$. In fact, if $f \in C(X)$, then

$$\begin{aligned} \nu(f) &= \int f \, d\nu = \int \mu_y(f) \, d\nu(y) \\ &= \int_T \mu_y(f) \, d\nu(y) + \int_{X \setminus T} \mu_y(f) \, d\nu(y) \end{aligned}$$

Since T is at most countably infinite and the support of $\mu_y (y \in T)$ is an ASPO, the first term above may be written as

$$\sum_{y \in T} \mu_y(f) \, \nu(y) = \sum_{y \in T}{}' \mu_y(f) \, \nu(\text{support } \mu_y)$$

If $\nu(X \setminus T) \neq 0$, define $L: C(X) \to R$ by

$$L(g) = 1/\nu(X \setminus T) \cdot \int \chi_{X \setminus T}(y) \, \mu_y(g) \, d\nu(y)$$

By the Riesz representation theorem[5] there exists a unique measure $\pi \in M(X)$ such that $L(g) = \pi(g)\ \forall g \in C(X)$. From the definition of L one easily concludes that $\pi \in I(\mathscr{A})$.

The following correspondences show that the decomposition is valid: $q_y = \nu(\text{support } \mu_y)$; if $\nu(X \setminus T) \neq 0$, then $\pi = \sigma$ and $q = \nu(X \setminus T)$; if $\nu(X \setminus T) = 0$, then $q = 0$.

Since $\nu(T) = \sum'_{y \in T} \nu(\text{support } \mu_y) > 0$, there is $w \in T$ such that $q_w = r > 0$. One can assume, without loss of generality, that $\mu_w = \delta_p$ for some $p \in X$.

Recall that the support of δ_p is an ASPO; therefore G_{δ_p} is nonempty and there is a neighborhood V of p such that $V \subset G_{\delta_p}$.

Lemma 9. $G_\nu \subset G_{\delta_p}$.

Proof. Let $x \in G_\nu$. Now we have the decomposition

$$\nu = r\delta_p + (1 - r)\rho$$

where $\rho \in I(\mathscr{A})$. Pick $\varepsilon > 0$ such that $\overline{B(p, 2\varepsilon)} \subset V$, where $B(p, 2\varepsilon)$ denotes the open ball with center p and radius 2ε, and the bar represents the closure operation.

Let u_ε be an Urysohn function such that

$$u_\varepsilon(z) = \begin{cases} 1 & \text{if } z \in \overline{B(p, \varepsilon)} \\ 0 & \text{if } z \in X \setminus B(p, 2\varepsilon) \\ 0 < u_\varepsilon < 1 & \text{otherwise} \end{cases}$$

Since u_ε is continuous, one has

$$\liminf_{n \to \infty} \frac{1}{n} (S_n \chi_{B(p, 2\varepsilon)})(x) \geqslant \lim_{n \to \infty} \frac{1}{n} (S_n u_\varepsilon)(x) = \int u_\varepsilon \, d\nu$$

By the above decomposition

$$\int u_\varepsilon \, d\nu = r \int u_\varepsilon \, d\delta_p + (1 - r) \int u_\varepsilon \, d\rho \geqslant r > 0$$

Hence

$$\liminf_{n \to \infty} \frac{1}{n} (S_n \chi_{B(p, 2\varepsilon)})(x) > 0$$

and therefore there is a $k \in N$ such that $\mathscr{A}^k(x) \in B(p, 2\varepsilon) \subset V \subset G_{\delta_p}$. Since $\bigcup_{j=1}^{\infty} \mathscr{A}^{-j}(V) \subset G_{\delta_p}$, one has $G_\nu \subset G_{\delta_p}$. ∎

To conclude the proof of the theorem, it is enough to observe that since $G_\nu \neq \varnothing$, we obtain $\nu = \delta_p$. Let $x \in (G_\nu \cap G_{\delta_p})$; then

$$\int f\, d\nu = \lim_{n \to \infty} \frac{1}{n} (S_n f)(x) = \int f\, d\delta_p \qquad \forall f \in C(X)$$

Therefore $\nu = \delta_p$, i.e., ν is an ASPO.

5. FLOWS

In this Section, I consider the case of flows. Let ψ_t be a flow generated by a $C^{1+\alpha}$ vector field on X. The definitions, results, and proofs for maps given in Section 2 are analogous for ψ_t. I shall consider an attractive fixed point for ψ_t as an ASPO of period zero. For the generalization of Theorem 6 for flows the following result due to Campanino[2] is needed.

Lemma 5A. Let ψ_t be as above and μ be an ergodic probability measure for ψ_t.

(i) If $\lambda^1 < 0$ μ-a.e., then the support of μ is an ASPO of period zero.

(ii) If $\lambda^2 < 0$ μ-a.e. and the vector field does not vanish for some point in the support of μ, then the support of μ is an ASPO of period greater than zero.

Now I state and sketch the proof of the result corresponding to Theorem 6 for flows. The proofs of the Riesz representation theorem and the ergodic decomposition theorem (Lemma 7) for flows can be found in ref. 7.

Theorem 6A. Let ν be an invariant measure for the flow $\psi_t: X \to X$ generated by a $C^{1+\alpha}$ vector field such that $G_\nu \neq \varnothing$.

(j) If there is a measurable set $F \subset X$ of nonzero ν-measure such that all CEs are less than zero at any point of F, then ν is ergodic and its support is an ASPO of period zero.

(jj) If there is a measurable set $F \subset X$ of nonzero ν-measure such that $\lambda^2 < 0$ at any point of F and the vector field does not vanish in F, then ν is ergodic and its support is an ASPO of period greater than zero.

Corollary. If instead of $G_\nu \neq \varnothing$ in Theorem 6A one assumes that ν *is an attractor for* $\mathscr{A}$, the same conclusions hold.

Proof of Theorem 6A. (j) The proof of Theorem 6A(j) is analogous to the proof of Theorem 6.

(jj) One has $\int_F \lambda^2 \, d\nu < 0$. By the ergodic decomposition of ν there is a set T with $\nu(T) > 0$ and $\int_F \lambda^2 \, d\mu_y < 0$ whenever $y \in T$ (recall that μ_y is ergodic).

Lemma 8A. The cardinality of T is at most countably infinite, and if $y \in T$, the support of μ_y is an ASPO of period greater than zero.

Proof. Since $\int_F \lambda^2 \, d\mu_y < 0$, one has $\lambda^2 < 0$ μ_y-a.e. and $\mu_y(F) > 0$; then there is a point in F that belongs to the support of μ_y (this result easily follows from the regularity of μ_y). By Lemma 5A(ii) the support of μ_y is an ASPO of period greater than zero.

The rest of the proofs of Lemma 8A and Theorem 6A(jj) are analogous to the proofs of Lemma 8 and Theorem 6, respectively. ∎

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